A POST-LAGRANGE/POINCARE LOOK AT CALCULUS

In his recent report on the calculus reform in the field, Tucker writes that "*the content of the new calculus may not differ that much from the old calculus. (...). Calculus may have a core of ideas and techniques that must remain invariant under any transformation..*" A couple of years into the "calculus initiative" and a feeling seems already to be developing that, crisis or no crisis, changes to the calculus are either much harder to be brought about than anticipated or perhaps, after all, not even possible. Adding "*This is depressing in a way, however. If contents were the problem, we would have a quick fix: change the textbooks*", Tucker then echoes what seems to be an emerging consensus: if there is any room for improvement, and it is by no means generally agreed that there is, e.g. Anton's talk in Louisville, then what seems to be neeeded is a new improved delivery system, preferably computerized.

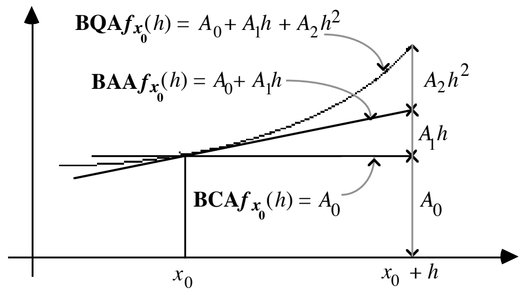
We disagree. As we see it, the contents are subordinated to our *conception* of the calculus (including differential equations), both as a theory and in terms of the questions it is intended to address and this is where the problem lies. The currently prevailing conception involves: **i.** an almost complete disregard of the *study* of functions (understood as in the French *Etude de function*) and **ii.** an obsession with calculating everything in closed form. That the conventional conception of the calculus is based on the notion of limit is a further irony given that is does not yield transparent descriptions of the bahaviour of functions and that it offers no algorithm to *find* limits.

We would argue that inasmuch as functions are the mathematical expression of "processes", a curriculum truly for "*just plain folks*" should take a geodesic course toward the study of Dynamical Sytems as this is how most functions appear in real life and almost *never* in closed form: the differential system $\dot{x} = f(x)$, is studied near x_0 by approximating $f(x)$ near x_0 by its Taylor approximation: either $f(x_0)\neq 0$ in which case we find that nothing of very much interest happens near x_0 or $f(x_0)=0$ in which case we approximate $f(x_0+h)$ by the next term, $f'(x_0)h$ etc.

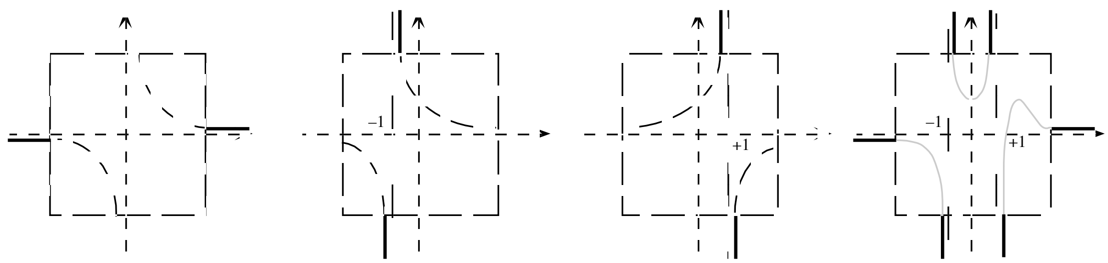
This then means that **i.** differential equations should be introduced as soon as possible, at least in particular cases, and in any case well before the integral calculus and **ii.** even polynomial and rational functions should be studied in a way that announces and prepares the way solutions of differential systems will be studied. In other words, we are advocating the study of functions by way of asymptotic expansions.

With students having only a background in basic algebra, and starting with power functions as *gauge functions*, we are able to study affine, quadratic, polynomial and rational functions as locally approximately (Laurent–) polynomial functions during the first semester. We introduce almost immediately the idea of differential equation. For example, the "point-slope formula" for a straight line appears as solution of the differential equation $f'(x) = m$ with initial condition $f(x_0) = y_0$. During the second semester, we discover new behaviours, particularly at ∞, with algebraic functions defined as solutions of algebraic equations and "elementary" functions as solutions of differential equations. We obtain approximate solutions by the method of indeterminate coefficients and after showing that they have properties such as the addition formula we assert them to be approximations of exact solutions. After the exponential function has been characterized near infinity as exceeding all power functions, it is included among the gauge functions «Dieudonné, 1968/1971 #52». This could be followed by a global study as in «Lang, 1976 #18» or «Finney, 1984 #16». If desired, *sided* limits can be obtained algorithmically with l'Hospital rule made unnecessary.

Local best polynomial approximations are easily visualized because their graphs,among which the tangent, (see «Bivins, 1986 #32» which won its author both the G. Polya and the Merten M. M. Hasse prizes) are osculating lines. Given $f(x_0+h) = A_0 + A_1h + A_2h^2 + ...$, we construct:



As an example, the function $f(x) = \dfrac{x-2}{x^2-1}$ can be graphed as follows: at ∞, $f(x) = \dfrac{x+\ldots}{x^2+\ldots}$ and by division in *de*scending powers $f(x) = \dfrac{1}{x} + \ldots$. At the poles, $f(-1+h) = \dfrac{-3+\ldots}{-2h+\ldots}$ and by division in *as*cending powers $f(-1+h) = \dfrac{3}{2h} +\ldots$. Similarly, $\boldsymbol{f}(1+h) = \dfrac{-1+\ldots}{2h+\ldots} = \dfrac{-1}{2h} +\ldots$. We then sketch the local graphs and interpolate smoothly:



$\boldsymbol{f}_\infty(x) = \dfrac{1}{x}$ $\quad$ $\boldsymbol{f}_{-1}(h) = \dfrac{3}{2h}$ $\quad$ $\boldsymbol{f}_{+1}(h) = -\dfrac{1}{2h}$ $\quad$ "Essential" graph of $\boldsymbol{f}$

Qualitatively, we define the *sign* of a function near x_0, (+,+), (+,–), (–,+), (–,–), that is the way values near x_0 differs from 0, we compare the first non-zero approximation to the Zero Function. To define the *variation* near x_0, (Ω,Ω), (Ω,œ), (œ,Ω), (œ,œ), that is the way values near x_0 differ from the value at x_0, and the *optimization* near x_0, (max,max), (max,min), (min,max), (min,min), that is the way the value at x_0 differ from values near x_0, we compare the first best non-constant approximation to the Best Constant Approximation. To define the *concavity* near x_0, ($%,$%), ($%,^&), (^&,$%), (^&,^&), that is the way the slope near x_0 differs from the slope at x_0, we compare the first best non-affine approximation to the Best Affine Approximation. This yield a very exporessive language to discuss the local behaviour of functions. For instance, the "second derivative test" is seen as follows: Given, we get Sign $f''(x_0+h)$ from the sign of $f''(x_0)$ or, if $f''(x_0) = 0$, directly. In either case, this gives Concavity $f(x_0+h)$ and, if x_0 is a critical point, that it is an extreme.

Systematically enlarging the universe of functions then lends itself particularly well to gradual *generalizations* because there is no technical difficulty to obscure the concepts being discussed. Consider, for instance, how what we call the Zero Theorem generalizes. In a universe which consists of only the zero function, the Zero Theorem reads: a function is zero everywhere; in the universe of constant functions, a function is either zero everywhere or nowhere; in the universe of affine functions, a function is either zero everywhere or nowhere or at exactly one point; in the universe of quadratic functions, we must take into account the multiplicity of the zero and look at Sign $f(x_0+h)$.

Quantitatively, we define the (Peano) derivatives from the coefficients in the best polynomial approximations. Thus low degree polynomial functions induce a qualitative hierarchy «Gleason, 1967 #34»: continuous functions are locally approximately constant, differentiable functions are locally approximately affine, etc.

For the third semester, we plan on developing a course on dynamical systems as an alternative to the integral calculus much better suited to the needs of students intending to pursue a career in the sciences but not, a priori, in mathematics, physics or engineering. But even to motivate the relation between the antiderivative and the definite integral, we would rather follow Picard «Picard, 1901 #30»:

"*Integral Calculus was born the day one asked the question: given $f(x)$, does there exist a function whose derivative is $f(x)$, in other words a function which satisfies*

$$\frac{dy}{dx} = f(x) \tag{1}$$

This question was at first answered by a geometrical interpretation which, even though it had no value in itself, helped greatly with the solution of the problem: One graphs first the function f then one considers the area bounded by this curve, the x-axis and two parallels to the y-axis, one fixed, the other one variable. One then shows that the area, considered as a function of the x-intercept x of the second parallel is a function of x having $f(x)$ as derivative. It is clear that, unless one assumes that the notion of area is given, the problem has not been solved rigorously. We assume f continuous. The following considerations lead naturally to the algebraic expression which plays a fundamental role in the Integral Calculus. Assume, for a moment, the existence of a function y satisfying (1), with $y(a) = y_0$ and $y(b) = Y$. Subdivide the interval $[a,b]$ in n intervals and let $x_1, x_2, \ldots, x_{n-1}$, be the x-coordinate of the subdividing points. Let $y_1, y_2, \ldots, y_{n-1}$ be the corresponding values for y. If the interval x_1-a is small enough, the quotient $\frac{y_1-y_0}{x_1-a}$ is very close to $f(a)$ and we have the following equations which hold only approximately:

$$\begin{aligned} y_1-y_0 &= (x_1-a)f(a) \\ y_2-y_1 &= (x_2-x_1)f(x_1) \\ &\;\;\vdots \\ Y-y_{n-1} &= (b-x_{n-1})f(x_{n-1}) \end{aligned}$$

Adding them up, we obtain:

$$Y-y_0 = (x_1-a)f(a) + (x_2-x_1)f(x_1) + \ldots + (b-x_{n-1})f(x_{n-1})$$

This holds only approximately but, hopefully, the approximation will get better and better as the number of intervals increases and the length of each one goes to 0. We are thus led, given a continuous function f to study the sum $(x_1-a)f(a) + (x_2-x_1)f(x_1) + \ldots + (x_2-x_1)f(x_1)$."

We believe that it is only in such a way that we will be really able to bring about the changes that, deep in our heart, we all know are indispensible. Unfortunately, other than in subjects dealing technically with asymptotic expansions such as differential topology, number theory, etc and other than ... in French, there is an almost complete lack of literature on the subject. Then, many will surely be the letters to the editor to say about some or the other of the above, and to paraphrase «Dudley, 1988 #44»,: "Evidently, the authors are unaware that ..." or to say that this approach emphatically cannot work "because it lacks the intuitive notion of the tangent being approached by a secant " or "because we have enough trouble teaching Taylor series". "*Eppur si muove*".